International Journal of Theoretical Physics, Vol. 4, No. 2 (1971), pp. 101–115

Local Quantum Entropy

J. G. GILSON

Department of Mathematics,
Queen Mary College, London

Received: 12 September, 1970

Abstract

It is shown that it is possible to define 'true local entropy' in velocity space, in an approximate version of the two-fluid formulation of quantum theory introduced by the present author in earlier papers. Using this definition, it is then shown that it is possible to define finite forms for total entropy at all points in configuration space. This important step is achieved by the introduction of a 'responding' velocity space. The use of a basis system which responds to occupation number density, makes possible a clear separation of the statistics and the dynamics of the underlying quantum process, and also makes possible the unambiguous use of certain divergent and oscillatory integrals.

1. *Introduction*

In a series of earlier papers (Gilson, 1968a, b, c, 1969a, b, c) the present author has made much use of various generalisations of Wigner's phase space distributions to show that Schrödinger quantum mechanics can be expressed in forms surprisingly close to classical fluid theory and thermodynamics. It appears from this work that a completely 'classical' formulation of quantum theory is possible. However, one does encounter some severe mathematical difficulties. One of these difficulties lies in the appearance of divergent integrals which up to now one has only been able to work with on a very intuitional basis. This difficulty is due to the often required need in quantum investigations to analytically continue functions such as the Gaussian, $\exp(-x)^2$, into regions of complex x, when the grossly divergent factor, $\exp(+y^2)$, appears in integrals. It turns out, as will be shown, that for our main purpose, which is to define local quantum entropy, this difficulty can be unambiguously overcome.

Another difficulty which is troublesome is the fact that, in a local theory, functions will generally depend on more variables than they do in the global type of theory, of which the Fourier transformed Schrödinger equation formulation is an example. In the transformed Schrödinger equation, momentum appears as a simple parameter, p, whereas in the local theory momentum appears as a function of x and t. In functions describing other properties, these local complexities may be great. This has the consequence that it is difficult to specify in detail general characteristics of such functions and of integrals which involve them. However, this type of difficulty is more technical than fundamental, and usually can be overcome. In fact, it is the greater complexity of texture which is possible in a local theory which gives such a theory a richness and a potentiality in excess of that possessed by conventional quantum theory.

2. *The Joint Fluid Distribution*

Expressed as a mass distribution over a two-fluid velocity space (v_1, v_2), a function which contains all the information (Gilson, 1969b, c) which can be derived from the usual Schrödinger equation is

$$m(x, t|v_1, v_2) = \rho^{-1} \frac{m_0^3}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \psi^*\left(x - \frac{\hbar\tau}{2}, t\right) \psi\left(x + \frac{\hbar\tau^*}{2}, t\right)$$

$$\times \exp[-i(v_1 \tau_1 + v_2 \tau_2) m_0] \, d\tau_1 \, d\tau_2 \qquad (2.1)$$

where

$$\tau = \tau_1 + i\tau_2 \qquad (2.2)$$

and

$$\rho = \psi^*(x, t) \psi(x, t) \qquad (2.3)$$

It is not necessary to specify that the function $\psi(x, t)$ is a solution to the Schrödinger equation. Rather, one can impose the condition that the two fluids are in a state of local thermal equilibrium (Gilson, 1969b) defined by

$$\mu(\text{solute}) + \mu(\text{solvent}) = 0 \qquad (2.4)$$

where the μ's are thermal energies associated with the two fluids. In general, the function $m(x, t|v_1, v_2)$ is a complicated function of the four variables x, t, v_1 and v_2. It is also a functional of $\psi(x, t)$. Because there is much scope for the functional form of $\psi(x, t)$, even though it is restricted to being a solution to the Schrödinger equation as a consequence of (2.4), any general specification of the properties of $m(x, t|v_1 v_2)$ is of considerable technical difficulty. The main type of mathematical pathology exhibited by (2.1) is already present in an approximation to (2.1) which we shall now discuss in detail. We must further remark that $m(x, t|v_1, v_2)$ is not a function which is positive everywhere. It can assume negative values in some parts of v_1, v_2 space (Gilson, 1969c). If we expand $\log \psi(x + \hbar\tau^*/2, t)$ and $\log \psi^*(x - \hbar\tau/2, t)$ as far as terms quadratic in τ_1 and τ_2, we get the following approximation to (2.1)

$$m \simeq \frac{m_0^3}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[\exp\left\{i(\tau_1 \phi_1' + \tau_2 \phi_2') - \frac{\hbar}{4}(\tau_1^2 - \tau_2^2) \phi_2''\right.\right.$$

$$\left.\left. + \frac{\hbar}{2} \tau_1 \tau_2 \phi_1'' - i m_0(\tau_1 v_1 + \tau_2 v_2)\right]\right\} d\tau_1 \, d\tau_2 \qquad (2.5)$$

Here
$$\phi = \phi_1 + i\phi_2 = \frac{\hbar}{i} \log \psi(x, t) \tag{2.6}$$

and the primes denote differentiations with respect to x. At this point, we encounter an interesting problem. One of the τ integrations in (2.5) can always be done leaving a possibly divergent integral in the other τ variable. That is, the τ_1 integral can be carried out where $\phi_2'' > 0$ and then the τ_2 integration involves the divergent exponent $\exp(\tau_2^2 \phi_2''/4)$, and vice versa for $\phi_2'' < 0$. However, it is possible to get some idea on how to proceed by pretending that both integrations in (2.5) can be done simultaneously. If we do, temporarily, take this point of view and make use of the formula,

$$\int_{-\infty}^{+\infty} \exp[i(-\lambda\tau^2 + k\tau)\, d\tau] = (\pi/\lambda i)^{1/2} \exp\left(+\frac{ik^2}{4\lambda}\right) \tag{2.7}$$

to perform the integrations in τ_1 and τ_2 successively, we get the following incorrect but instructive evaluation of (2.5).

$$m \mathrel{\hat{=}} \frac{m_0{}^3}{\pi i(\epsilon_1{}^2 + \epsilon_2{}^2)^{1/2}} \exp\{a m_0{}^2(u_2{}^2 - u_1{}^2) + b m_0{}^2 u_1 u_2\} \tag{2.8}$$

where

$$a = \epsilon_2/(\epsilon_1{}^2 + \epsilon_2{}^2) \tag{2.9}$$
$$b = 2\epsilon_2/(\epsilon_1{}^2 + \epsilon_2{}^2)$$
$$\epsilon = \epsilon_1 + i\epsilon_2 = \phi''(x, t) \tag{2.10}$$

and

$$u = u_1 + iu_2 = v - \phi'(x, t)/m_0 \tag{2.11}$$

is the complex thermal velocity relative to the mean complex velocity. The complex expressions (2.10) and (2.11) are here merely a convenient shorthand for combining pairs of expressions. Expression (2.8) is incorrect in at least two respects. These are that a factor, $i = \sqrt{-1}$, has appeared in what should be a real function, and the divergence mentioned earlier has been bypassed. However, expression (2.8) does suggest an interesting conjecture, which we shall prove to be correct. The conjecture is that, in spite of the divergence problem, local velocity space entropy can be defined and as far as dependence on u_1 and u_2 is concerned it is essentially given by an expression of the form

$$\sigma(u_1, u_2) = \frac{1}{2kT}[u_2{}^2 - u_1{}^2 + x u_1 u_2] \tag{2.12}$$

Actually, it turns out that (2.12) might more appropriately be called the relative local entropy of fluid '2' with respect to fluid '1'. It is necessary that we should be able to define local entropy in velocity space, if we are to employ a classical definition for total entropy such as

$$H = \int \int \sigma(u_1, u_2) f(u_1, u_2)\, du_1\, du_2 \tag{2.13}$$

This would be equivalent to Boltzmann's H function.

3. *The Separate Fluid Distributions*

The integrand in expression (2.5) could be factorised into two parts, one only involving fluid '1' variables, and the other only involving fluid '2' variables, if the $\tau_1 \tau_2$ term were not present in the exponent integrand. Thus, if we could neglect the $\tau_1 \tau_2$ term, the function m itself would factorise into two parts, each only involving one of the fluids. It is clear that such a situation would correspond to the usual classical case when the product of two separate distributions gives the joint two-fluid distribution. The introduction of expression (2.1), and consequently the derivation of (2.5), can only be said to have had a certain *ad hoc* justification, and so we are really free to take as basis for this work forms other than (2.1), provided these other possibilities are as suitable, or preferably better, than our original choice. A better choice of starting point would, in the light of experience, be the two separate distributions implied by (2.5), if the $\tau_1 \tau_2$ term were neglected. Expression (2.5) could then be reserved to describe the more complex situation when the additional interaction described by the $\tau_1 \tau_2$ term is present. In fact, from the point of view of the limited objectives of this paper which are centred around defining some appropriate forms for entropy, we can disregard the $\tau_1 \tau_2$ term for a rather specific reason. This reason is that the $\tau_1 \tau_2$ term in (2.5) leads directly to the $x u_1 u_2$ term in (2.12) and if a term such as this is included in the local velocity space entropy it always averages to zero. This is because of the two results (Gilson, 1969b) contained in the real and imaginary parts of

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (v - \phi'/m_0)\, m(x,t|v_1,v_2)\, dv_1\, dv_2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u m(x,t|v_1,v_2)\, dv_1\, dv_2 = 0 \tag{3.1}$$

where (2.11) has been used. Thus a term depending linearly on u_1 or u_2 will not contribute to the total configuration space entropy at any point (x, t). As has been said, we are here only concerned with finding a suitable definition for entropy and with some of the consequences which will follow from such a definition. Thus, for these purposes, provided that we can find a justification for the square term part of (2.12) appearing in the velocity space definition of entropy, we can disregard the product term in the joint distribution (2.5) and take the two separate velocity space distributions suggested by (2.5) as the basis of our work. Let us denote these distributions by R_1 and R_2.

$$R_1 = \frac{A_1(\alpha_1)}{2\pi} \int_{-\alpha_1}^{+\alpha_1} \exp(-\lambda \tau_1^2 - i m_0 u_1 \tau_1)\, d\tau_1 \tag{3.2}$$

$$R_2 = \frac{A_2(\alpha_2)}{2\pi} \int_{-\alpha_2}^{+\alpha_2} \exp(+\lambda \tau_2^2 - i m_0 u_2 \tau_2)\, d\tau_2 \tag{3.3}$$

where

$$\lambda = \frac{\hbar}{4}\phi_2'' \tag{3.4}$$

A_1 and A_2 are functions of α_1 and α_2 and possibly of x and t. They have been introduced in anticipation of the possibility that (3.2) and (3.3) are not necessarily normalised for all x and t. The infinite limits of integration which appear in (2.5) have now been replaced by the finite values $\pm\alpha_1$ and $\pm\alpha_2$ in order that we may work at least initially with convergent and well defined integrals.

In (3.2) and (3.3), λ is a function of x and t, and is not necessarily positive for all x and t. Thus, according to whether λ is positive or negative, so the integral in (3.3) or the integral in (3.2) will be infinite and oscillatory in the limit of α becoming infinite.

4. *A Physical Basis*

We shall now suggest a basis for the following work in terms analogous to the equilibrium described by the equality of the chemical potentials for two species in contact. We shall assume, in line with earlier work (Gilson, 1969b), that the physical system underlying the processes which are usually described by the Schrödinger equation is a two-constituent system. Elsewhere we have called these two constituents ‘the solute’ and ‘the solvent’. This is a convenient terminology to retain. The solute is the constituent which most nearly approaches a simple classical fluid in character. The solvent is to be regarded more as a background fluid against which the solute is described. However, it can be seen that as the function $\lambda(x,t)$ can change its sign the two constituents can interchange their roles. Let us first consider the solute, and make the firm assumption that it can be described by a local form of Maxwell distribution up to the order of approximation at which we shall work. Thus let its number concentration in velocity space be

$$f_1(v_1 - \bar{v}_1(x,t)) = \left(\frac{m_0}{2\pi k|T(x,t)|}\right)^{1/2} \exp\left(-\frac{m_0(v_1 - \bar{v}_1(x,t))^2}{2kT(x,t)}\right) \tag{4.1}$$

Before proceeding, we should draw the reader’s attention to a number of features of (4.1). First, the temperature, $T(x,t)$, is assumed to vary with position x and time t. Secondly, the velocity, $u(=v_1 - \bar{v}_1)$, is the thermal velocity relative to the mean fluid velocity, $\bar{v}_1$. Thirdly, we have taken $|T(x,t)|$ in the square root rather than just $T(x,t)$. This last feature will clearly make no difference where $T(x,t)$ remains positive. However, we are going to allow T to assume negative values and the modulus is there in order to avoid the appearance of a $\sqrt{-1}$. Where $T > 0$, we shall have the usual normalisation condition

$$\int_{-\infty}^{+\infty} f_1(u_1)\,du_1 = 1 \tag{4.2}$$

let us now define a function, $\mu(u,T)$, of two functions, $g(u)$ and $T(x,t)$, as follows

$$\mu(u,T) = \log g(u) + \frac{mu^2}{2kT} \tag{4.3}$$

We now impose on $\mu(u,T)$ the further condition that it is to be independent of u. That is to say, we take $\mu(u,T)$ to be an invariant in thermal velocity space. Under this condition, $g(u)$ must be the $f(u)$ given by (4.1), apart from a multiplicative function of T only. Thus μ can be regarded as playing a role rather like the chemical potential and, in fact, expressing (by its invariance in velocity space) the equilibrium of the contributions from the various thermal velocities. We can now take this equilibrium idea further. If T changes sign in (4.3), then a corresponding change of sign of T in (4.1) will give another $g(u)$, which still leaves $\mu(u,T)$ with the same constant value. Thus, besides the invariance of μ in velocity space expressing the equilibrium between all the velocity constituents, this invariance also expresses another possible equilibrium between a fluid with the distribution (4.1) and another fluid with the distribution

$$f_2(v_2 - v_2(x,t)) = \left(\frac{m_0}{2\pi k|T(x,t)|}\right)^{1/2} \exp\left(+\frac{m_0(v_2 - \bar{v}_2(x,t))^2}{2kT(x,t)}\right) \tag{4.4}$$

say. We have anticipated that (4.4) represents our solvent by using the subscript '2'.

It is clear that for $f_2(u_2)$ we have not got a normalisation condition like (4.2). However, we will show later that there is an equivalent condition. Thus, we shall take the two distributions (4.1) and (4.4) as the basis for the description of the two fluids underlying the quantum processes. These two distributions are linked by being dependent on the common function $T(x,t)$ and also by a more indirect connection between the functions $\bar{v}_1(x,t)$ and $\bar{v}_2(x,t)$. At this point we can make the second firm assumption that the two distributions (4.1) and (4.4) contain essentially all the statistics in the theory. The expression (4.1) for $f_1(u_1)$ is most satisfactory in this respect (for $T > 0$) because it is very convergent as $u_1 \to \pm\infty$ and it can obviously be regarded as a pure probability density in velocity space. A similar remark applies to $f_2(u_2)$ for $T < 0$. The expression (4.4) for $f_2(u_2)$ for $T > 0$ (and indeed f_1 for $T < 0$) presents more difficulties. It cannot, as it stands, be regarded as a pure probability contribution because it diverges as $u_2 \to \pm\infty$ and it is clearly not normalisable. There is no reason, however, why it should not be regarded as giving the occupation number density for the various velocity states in the case of fluid '2'. Thus it is convenient to regard both (4.1) and (4.4) as representing the numbers of 'particles' present in the velocity ranges δu_1 and δu_2. Having assumed that $f_1(u_1)$ and $f_2(u_2)$ contain all the statistical features of our thermal velocity space description, the question arises as to what use are we to put this statistical information. What shall we average? We shall show that averages of a certain dynamical characteristic (which we shall call 'r') make possible the link with conventional quantum theory.

5. *Velocity Space Response*

This step of regarding all the statistics as relegated to the two Maxwell-like distributions (4.1) and (4.4) is an important one. However, we clearly have not yet got a suitable basis for quantum mechanics, because we have yet to relate (4.1) and (4.4) to the type of 'distribution' (2.1) which arises in the two-fluid theory and which we have proved elsewhere (Gilson, 1969b) corresponds to quantum mechanics. The fact that we now have to make this connection holds the implication that the quantum mechanical orientated distributions are not just pure statistics. They are, in fact, statistics weighted with some dynamical feature, and this we shall demonstrate. In earlier work (Gilson, 1969c), we used a mass distribution in order to clarify one of the anomalous features of the quantum-orientated distributions. In particular, the fact that there is a place in dynamics for negative mass enabled us to make a correlation between negative mass states and negative 'probability'. We shall not need to continue to assume that our statistics is mass weighted for the purposes of this paper. We can usefully talk in rather more general terms and leave open the question of whether the weighting of our statistics should be mass, charge spin, etc., or indeed some new physical parameter. Thus we shall now assume that we are studying a statistical two-fluid system which requires for its complete description an additional physical parameter over and above those usually required for simple classical fluid systems. Let us denote this new parameter by the letter 'r'. In particular, r will have to play the role of representing the non-statistical aspects of the fluid system in velocity space. Usually a statistical system is described against a static geometrical background. In order to bring a hyper-statistical element into our fluid scheme we shall now assume that our system is naturally described against a non-static geometrical background and it will be the parameter r which describes this non-static aspect of the geometry. We shall use a thermal velocity (or phase) space which under some circumstances responds (r) to the occupancy of its states. The parameter r is to denote this response. In the case of distribution (4.1) for $T > 0$ no response is necessary because of its simple classical form. In the case of distribution (4.4) for $T > 0$, which can be regarded as, in some sense, overloading velocity space, a response must be allowed. If it were not allowed, unlimited numbers of particles would be effective at large velocities and this would cause a conceptual breakdown of the theory (or indeed a physical breakdown of the system). The purpose, then, of r is to make physically tolerable those statistical situations which would otherwise not be allowable classically but which seem essential in the quantum domain. In purely mathematical terms r can be regarded as a measure defined on velocity space.

As we have said before, the statistics is to be contained in the two

Maxwell-like distributions (4.1) and (4.4), while the availability of states of 'r' in velocity space is to be outside the statistics, and is thus the source of what often has seemed to be the statistical peculiarities of quantum mechanics (Suppes, 1963). Let us now consider the functional form we might reasonably expect the response function $r(u)$ to take.

Our thermal velocity space is now to respond to some 'abnormal' states of occupancy. Thus our velocity space is dynamic. We can assume that the fact that the velocity space is being occupied causes a response of self modification which takes the form of making available bands of possible positive and negative response states. In the mass distribution interpretation these states would be the location of positive and negative mass. These states are then occupied in a specific way. These two processes take place simultaneously and are a self-adjusting and stabilisation process for the whole system. A reasonable physical assumption is that the response is a locally 'felt' effect. Thus in the first instance it will depend on position in velocity space and on the local temperature which is non-uniform in this scheme. Thus we shall assume that r is at least a function of u and T. One would also expect that the response at velocity u and temperature T would depend on the complete information about occupancy throughout velocity space. Thus $r(u,T)$ will also be a 'functional' of the occupancy functions (4.1) and (4.4). Let us generally denote the occupation number functions by $f(u)$. The connection with conventional Schrödinger quantum mechanics will be via a relation of the form

$$R(u,T) = f(u)\, r(u,T) \tag{5.1}$$

where $R(u,T)$ now takes the place of the mass distribution (2.1) used in earlier work and $f(u)$ is either (4.1) or (4.2), or possibly other states of occupancy. As remarked earlier, it has been noticed (Suppes, 1963) for years that, statistically speaking, quantum mechanics is a queer theory. This shows up particularly in attempts to give velocity (or phase) space versions of the theory (Wigner, 1932; Moyal, 1949; Bartlett & Moyal, 1949; Leaf, 1968). One of the oddities is that it seems impossible to avoid the introduction of phase (or velocity) space distributions which have negative regions. It will be shown in this paper that when we have separated the statistics and the dynamics by making use of the r property such negative contributions arise from r and naturally negative 'probabilities' do not occur. Let us return to the question of the functional dependence of $r(u,T)$ on $f(u)$. A reasonable choice for this dependence is

$$r(u,T,[f]) = \int_{-\infty}^{u} F(f(u'))\, g_1(u')\, du' + \int_{u}^{+\infty} F(f(u'))\, g_2(u')\, du' \tag{5.2}$$

$F(f(u))$ is an ordinary function of the function $f(u)$ and $g_1(u)$ and $g_2(u)$ are ordinary functions of u. The response, r, thus depends locally on u and also has a functional dependence on f. This global dependence on f is indicated by the square brackets in r. The integration range in (5.2) has been purposely split into the two parts so as to achieve the two types of dependence in the simplest possible way.

A natural symmetry to be expected from $r(u)$ is

$$r(u)=r(-u) \tag{5.3}$$

This is the simple physical requirement that nature cannot distinguish between positive and negative thermal velocities. Thus, after a change of variable, we get from (5.2) and (5.3) the condition

$$\int_{-\infty}^{u} F(f(u'))\,g_1(u')\,du' + \int_{u}^{\infty} F(f(u'))\,g_2(u')\,du'$$

$$= \int_{-\infty}^{u} F(f(-u'))\,g_2(-u')\,du' + \int_{u}^{\infty} F(f(u'))\,g_1(-u')\,du' \tag{5.4}$$

If $f(u)$ is one of the even functions of u, (4.1) or (4.4) and provided $F(f)$ is assumed to be a polynomial function of f, then (5.4) suggests that

$$g_1(u)=g_2(-u) \tag{5.5}$$

That $F(f)$ is assumed to be no more complicated than a polynomial in f, can be regarded as a simplicity postulate. More involved theories could be developed on the basis of alternative assumptions for the form of $F(f)$.

If (5.5) is taken to apply, then the two parts of (5.2) can be combined together to give

$$r(u,T,[f]) = 2\int_{u}^{\infty} F(f(u'))\,g_2(u')\,du' + \int_{-u}^{+u} F(f(u'))\,g_2(u')\,du' \tag{5.6}$$

From (5.6), we see that, if $g_2(u)$ is taken to be an odd function of u, another simplification is possible. Thus we shall take

$$g_2(u) = -g_2(-u) \tag{5.7}$$

and so reduce the expression (5.6) for r to the form

$$r(u,T,[f]) = 2\int_{u}^{\infty} F(f(u'))\,g_2(u')\,du' \tag{5.8}$$

This author can see no obvious physical reason for assuming (5.7), unless it be something to do with causality. However, here we are looking for the simplest structure which is adequate, and so we shall use (5.7).

More general cases could be considered, but the form (5.8) which we have arrived at turns out to be sufficient for our purpose. At this stage it seems to be difficult to decide more closely the form of the functions F and g by general considerations. Thus we shall now turn to relating the form (5.8) to the more quantum orientated type of distributions (3.2) and (3.3) through the relation (5.1).

6. *Identification of* $r[f]$

In order to gain more information about the dependence of r on f and the function $g_2(u)$, we now make the identification implied by (3.2), (3.3) and (5.1). Thus we require that

$$\frac{A_1(\alpha_1)}{2}\int_{-\alpha_1}^{+\alpha_1} \exp(-\lambda\tau_1^2)\cos m_0 u_1 \tau_1\, d\tau_1 = f_1(u_1)\, r(u_1) \tag{6.1}$$

where $f(u_1)$ is given by (4.1) with $u_1 = v_1 - \bar{v}_1$. There is no difficulty when $\lambda > 0$. This is particularly so if α_1 is allowed to approach $+\infty$, because then the integral is easily evaluated. However, we shall not assume that $\lambda > 0$, but rather consider the case for general λ. If we differentiate r as given by (6.1) with respect to u_1, we get

$$\begin{aligned}\frac{\partial r}{\partial u} = -f_1^{-2}(u)\frac{\partial f}{\partial u}\int_{-\alpha}^{+\alpha} \exp(-\lambda\tau^2)\cos(m_0 u\tau)\, d\tau \,-\\ -f_1^{-1}(u)m_0\int_{-\alpha}^{+\alpha} \exp(-\lambda\tau^2)\sin(m_0 u\tau)\, d\tau\end{aligned} \tag{6.2}$$

$$\begin{aligned}= -f_1^{-1}(u)\left(-\frac{um_0}{kT}\right)\int_{-\alpha}^{+\alpha} \exp(-\lambda\tau^2)\cos(m_0 u\tau)\, d\tau \,-\\ -f_1^{-1}(u)\, m_0\left[\left(-\frac{1}{2\lambda}\right)\exp(-\lambda\tau^2)\sin(m_0 u\tau)\right]_{-\alpha}^{+\alpha}\\ +f_1^{-1}(u)\, m_0^2\, u\int_{-\alpha}^{+\alpha}\left(-\frac{1}{2\lambda}\right)\exp(-\lambda\tau^2)\cos(m_0 u\tau)\, d\tau\end{aligned} \tag{6.3}$$

where the second term in (6.2) has been integrated by parts and (4.1) has been used. If λ is positive, we know that the integral in (6.1) presents no difficulties and is easily evaluated when $\alpha_1 \to \infty$ to give

$$r(u_1) = \frac{A_1(\alpha_1)}{2\pi} \tag{6.4}$$

and

$$\lambda = m_0 kT/2 \tag{6.5}$$

This agrees with (6.3), which vanishes when (6.5) holds and $\alpha \to \infty$.

This is the classical situation when velocity space is static, and is characterised by the result (6.4) which says that the response is a constant on velocity space. However, when λ is negative things are distinctly different. If we return to our assumed form (5.8) for the response function and differentiate it with respect to u, we get

$$\frac{\partial r}{\partial u} = -2F(f_1(u))\,g_2(u) \tag{6.6}$$

We cannot now deduce (6.5) from the limiting form of our integral, but we can simply use (6.5) to define T in terms of λ. Thus from (6.3), we get

$$\frac{\partial r}{\partial u} = +\frac{2}{kT} f_1^{-1}(u) \exp(-m_0 kT\alpha^2) \sin m_0 u\alpha \tag{6.7}$$

This can be compared with (6.6) to give

$$F(f_1(u)) = f_1^{-1}(u) \tag{6.8}$$

and

$$g_2(u) = -\exp(-m_0 kT\alpha^2)(\sin m_0 u\alpha)(kT)^{-1} \tag{6.9}$$

Thus when λ is negative and consequently T is negative by (6.5) we cannot use the limit $\alpha \to \infty$ without (6.9) oscillating infinitely. However, we have now in (6.8) and (6.9) identified $F(f)$ and $g_2(u)$. It follows that the final form for the response function, functional is given by

$$r(u, T, [f]) = 2\int_u^\infty f_1^{-1}(u') \exp\left(-\frac{m_0 kT\alpha^2}{2}\right)(\sin m_0 u'\alpha)\, du (kT)^{-1} \tag{6.10}$$

after making use of the fact that $\sin(m_0 u\alpha)$ is an odd function of u. The two cases $\lambda \gtrless 0$ are both contained in (6.10) with the simplification arising for $T > 0$ and $\alpha \to \infty$ when we get the classical case, $r = \text{constant}$.

The same argument also applies for fluid '2'. In that case the λ in formula (6.10) appears with a plus sign, and instead of $f_1(u)$ the other function $f_2(u)$ now occurs in the functional r. Thus when one of the fluids is simple and classical, the other is of the overloading type, and thus causes a response like (6.10), with $\lambda < 0$. Because we cannot use the infinite α limit at all points (x, t), we must give some attention to the question of the range of integration of the α variables.

In order to allow the integration ranges to adjust to the variations of fluid character which occur where $T(x, t)$ changes sign, we shall adopt the following scheme.

Let

$$\begin{aligned} \beta_1 &= \alpha_1 \quad \text{where } T > 0 \\ &= \alpha_2 \quad \text{where } T < 0 \end{aligned}$$

and

$$\begin{aligned} \beta_2 &= \alpha_2 \quad \text{where } T > 0 \\ &= \alpha_1 \quad \text{where } T < 0 \end{aligned}$$

while α_1 and α_2 are both positive. These definitions are summarised in the formulae

$$\begin{aligned}\beta_1(T) &= \alpha_1\,\theta(T) + \alpha_2\,\theta(-T)\\ \beta_2(T) &= \alpha_2\,\theta(T) + \alpha_1\,\theta(-T)\end{aligned} \tag{6.11}$$

where $\theta(x)$ is the step function

$$\begin{aligned}\theta(x) &= 1 \qquad (x > 0)\\ &= 0 \qquad (x < 0)\end{aligned} \tag{6.12}$$

We can now redefine the integrals (3.2) and (3.3) with the more adaptable limits β_1 abd β_2. Thus in (3.2) and (3.3) we make the changes

$$\int_{-\alpha_1}^{+\alpha_1} \to \int_{-\beta_1(T)}^{+\beta_1(T)} \quad \text{and} \quad \int_{-\alpha_2}^{+\alpha_2} \to \int_{-\beta_2(T)}^{+\beta_2(T)} \tag{6.13}$$

and then α_1 can be allowed to approach infinity if it suits us, but α_2 must be kept finite. It is not difficult to show that, if this formalism is used to take averages of polynomials in the velocity variables, results do not depend at all critically on the values of the α integration limits. In fact, averages of powers of u_1 or u_2 always lead to expressions which depend on these limits through additive delta functions or derivatives of delta functions. Thus the α limits contribute by functions such as $\delta^{(n)}(\alpha)$, which is zero for $\alpha > 0$.

This particular feature was utilised in a provisional version of this formalism and in a somewhat different way in an earlier work (Gilson, 1969d). We now have a complete and unambiguous formalism which is adequate to cope with our objectives up to the order of approximation at which we have chosen to work. We shall now consider entropy.

7. *True and r Weighted Entropy*

The scheme developed in the previous sections gives a clear separation of the statistical and the dynamical aspects of the quantum orientated distributions (3.2) and (3.3). Thus the definitions for local velocity space entropies for the two fluids are immediate. They are

$$\sigma_1(u_1) = \frac{m_0\,u_1{}^2}{2kT} \tag{7.1}$$

and

$$\sigma_2(u_2) = \frac{m_0\,u_2{}^2}{2kT} \tag{7.2}$$

The actual signs of σ_1 and σ_2 being determined by the sign of the temperature, $T(x,t)$. Further, at points where $T(x,t)$ has a suitable sign total true entropies (H functions) are also straightforward. For example

$$H(x,t) = \int_{-\infty}^{+\infty} \sigma(u) \exp\left(\pm\frac{m_0 u^2}{2kT}\right) du (m_0/2\pi k|T|)^{1/2} \tag{7.3}$$

for one or other of the fluids, provided $\pm T < 0$.

In all our work we shall keep infinite integration ranges for velocity variables. Thus when $\pm T > 0$, integrals such as (7.3) will diverge, and it is at places where $\pm T(x,t) > 0$ that the r weighted entropy becomes important. Thus at such points we can define finite r weighted total entropies as

$$H_r(x,t) = \int_{-\infty}^{+\infty} \sigma(u) \exp\left(\pm\frac{m_0 u^2}{2kT}\right) r(u)\, du (m_0/2\pi k|T|)^{1/2} \tag{7.4}$$

and these will be well-defined convergent integrals and, indeed, they coincide with total true entropy at normal fluid positions. Further, the integrands in the $H_r(x,t)$ are directly correlated with the quantum orientated distributions (3.2) and (3.3) via the relations

$$\begin{aligned} &\exp\left(-\frac{m_0(v_1-\bar{v}_1)^2}{2kT}\right) r(v_1-\bar{v}_1, T, [f_1])\,(m_0/2\pi k|T|)^{1/2} \\ &\quad \doteq \rho^{-1}\frac{m_0}{2\pi}\int_{-\beta_1}^{+\beta_1} \psi^*\left(x-\frac{\hbar\tau_1}{2},t\right)\psi\left(x+\frac{\hbar\tau_1}{2},t\right)\exp(-iv_1\tau_1)\,d\tau_1 \end{aligned} \tag{7.5}$$

and

$$\begin{aligned} &\exp\left(+\frac{m_0(v_2-\bar{v}_2)^2}{2kT}\right) r(v_2-v_2, T, [f_2])\,(m_0/2\pi k|T|)^{1/2} \\ &\quad \doteq \rho^{-1}\frac{m_0}{2\pi}\int_{-\beta_2}^{+\beta_2} \psi^*\left(x-\frac{\hbar i\tau_2}{2},t\right)\psi\left(x-\frac{\hbar i\tau_2}{2},t\right)\exp(-iv_2\tau_2)\,d\tau_2 \end{aligned} \tag{7.6}$$

It is not difficult to confirm that the r functional enables us to write down an r-weighted normalisation condition equivalent to (4.2). This is

$$\int_{-\infty}^{+\infty} f_2(u_2)\, r(u_2)\, du_2 = 1 \tag{7.7}$$

If it is felt necessary to retain all infinite limits for the integrations, we can make use of the value of r at $u = 0$. The factors $A(\tau)$ introduced into equations (3.2) and (3.3) can then be exploited to define a local entropy constant in velocity space. Thus, if $s(u)$ is the then renormalised entropy, then $S(0) = 0$. This entropy vanishes for zero thermal velocity. Such a procedure could be regarded as a local form of application of Nernst's theorem.

It is, perhaps, not clear at this juncture how best to extend this theoretical structure. Such questions as to whether the τ integration limits should be kept finite or allowed to become infinite or, indeed, made to depend in more detail on other factors, may probably be answered in some wider context. Some further physical requirements might as well usefully be imposed on the structure. Clearly, the work in this area is very open ended and there are many options for further development. We shall now complete this section by discussing briefly the results of evaluating the r-weighted entropies for the two fluids at (x,t) in configuration space. These results are easily found to be

$$\int_{-\infty}^{+\infty} \sigma_1(u_1) f_1(u_1)\, r(u_1, T, [f_1])\, du_1 = +\tfrac{1}{2} \tag{7.8}$$

and

$$\int_{-\infty}^{+\infty} \sigma_2(u_2) f_2(u_2)\, r(u_2, T, [f_2])\, du_2 = -\tfrac{1}{2} \tag{7.9}$$

In this case of fluid '2', where $T(x,t) > 0$, the ordinary total true entropy is infinite and possibly of not much physical significance, and a similar remark applies to fluid '1' when $T(x,t) < 0$. From (7.8) and (7.9) we see that the sum of the total r-weighted fluid entropies is zero. This would seem to be another aspect of Nernst's theorem, with the implication that the local (in configuration space) 'entropy' of a pure quantum state is zero. Further, the relative [difference of (7.8) and (7.9)] local r-weighted entropy is a constant throughout configuration space and for all time. These features are strikingly reminiscent of the 'classical' two-fluid theory (London, 1954) for liquid helium, and suggests that the underlying quantum process is an interaction between a normal fluid and a superfluid.

However, the situation in this formalism for the quantum case is a somewhat different configuration of fluid and superfluid to that discussed by London (1954) for the liquid helium case. In the liquid helium situation one of the fluids can be considered to carry all the entropy. Here the total entropy of the two fluids is zero and the 'relative' motion carries all the entropy.

Conclusions

Most of the work in this paper is concerned with an approximation to Schrödinger quantum mechanics. However, it is this author's view that this particular approximation is of considerable significance and can give rather general information about the processes underlying quantum mechanics. It seems that this local description in velocity space can be brought very close to the classical Maxwell gas situation, in many respects, and can also be seen to have some very interesting similarities with superfluid theory. The complications which arise do not seem to be the result of essentially non-classical effects, but rather the consequence of properties of a substratum which is less simple than the well-studied classical systems. Within an approximate framework, we have shown that two types of entropy can be defined. It will, of course, be necessary to go beyond this approximation and this may not be an easy matter, but such further progress would appear to be of a technical nature without drastic philosophical difficulties. The works of Madelung (1926), Vigier (1954), Nelson (1966, 1967), Kershaw (1964), Bohm (1957) and Gilson (1968d) also contain work which is related to the ideas in this paper.

Acknowledgements

I am grateful to Professor M. S. Bartlett, F.R.S. for some early motivating discussions from which this work originated and to Dr. R. O. Davies for a number of recent useful discussions.

References

Bartlett, M. S. and Moyal, J. E. (1949). The Exact Transition Probabilities of Quantum Mechanical Oscillators calculated by the Phase-Space Method. *Proceedings of the Cambridge Philosophical Society. Mathematical and Physical Sciences*, **45**, 545.

Bohm, D. (1957). *Causality and Chance in Modern Physics*, Routledge & Kegan Paul, London.

Gilson, J. G. (1968a). On Stochastic Theories of Quantum Mechanics. *Proceedings of the Cambridge Philosophical Society. Mathematical and Physical Sciences*, **63**, 11, 1061.

Gilson, J. G. (1968b). Feynman Integral and Phase Space Probability. *Journal of Applied Probability*, Vol. 5 (August).

Gilson, J. G. (1968c). Expansion of Quantum Phase Space Probabilities into Classically Deterministic Parts. *Journal of the Institute of Mathematics*, **4**, 410.

Gilson, J. G. (1968d). Quantum Wave Functionals and Some Related Topics. *Zeitschrift für Naturforschung*, **23a**, 1452.

Gilson, J. G. (1969a). Quantum Probability Weighted Paths. *Acta physica Academiae scientiarum hungaricae*, **26**, 319.

Gilson, J. G.(1969b). Thermal Content of Quantum States. *Zeitschrift für Naturforschung*, **24a**, 198.

Gilson, J. G. (1969c). Relativistic Quantum Mass Distributions on Velocity Space. *Annales de l'Institut Henri Poincaré*, Vol. XI, No. 2, p. 239.

Gilson, J. G. (1969d). Quantum Velocity Space Distributions, *Zeitschrift für Naturforschung*, **24a**, 1431.

Kershaw, D. (1964). Theory of Hidden Variables. *Physical Review*, **136**, B1850.

Leaf, B. (1968). Weyl Transformations and the Classical Limit of Quantum Mechanics. *Journal of Mathematics and Physics*, Vol. 9, No. 1 (janvier 1968).

London, J. (1954). *Superfluids*, Vol. II, p. 126. John Wiley and Sons, London.

Madelung, E. (1926). Quanten Theorie in Hydrodynamischer Form. *Zeitschrift für Physik*. **40**, 332.

Moyal, J. E. (1949). Quantum Mechanics as a Statistical Theory. *Proceedings of the Cambridge Philosophical Society. Mathematical and Physical Sciences*, **45**, 99.

Nelson, E. (1966). Derivation of the Schrödinger equation from Newtonian Mechanics. *Physical Review*, **150**, 1079.

Nelson, E. (1967). *Dynamical Theories of Brownian Motion*, Princeton, University Press.

Suppes, P. (1963). *Delaware Seminar*, Vol. 2, Interscience.

Vigier, J. P. (1954). Model of the Causal Interpretation of Quantum Theory in terms of a Fluid with Irregular Fluctuations. *Physical Review*, **96**, 208.

Wigner, E. (1932). On the Quantum Correction for Thermodynamic Equilibrium. *Physical Review*, **40**, 749.